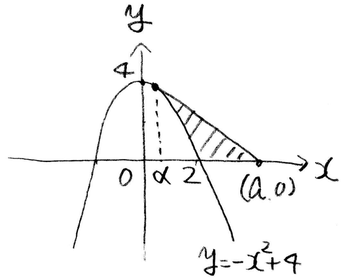
京大理系　1962前期　⑥



放物線上の点 $(\alpha, -\alpha^2+4)$ $(0<\alpha<2)$ における接線の方程式は

$y+\alpha^2-4=-2\alpha(x-\alpha)$, $y=-2\alpha x+\alpha^2+4$

これが $(a, 0)$ を通るから. $\alpha^2-2a\alpha+4=0$. $\alpha=a\pm\sqrt{a^2-4}$

$\alpha<a$ より $\alpha=a-\sqrt{a^2-4}$

求める面積をSとすると. $S=\int_\alpha^2(-2\alpha x+\alpha^2+\cancel{4}+x^2-\cancel{4})dx+(a-2)(-4\alpha+\alpha^2+4)$

$=\left[\frac{x^3}{3}-\cancel{2}\alpha\frac{x^2}{\cancel{2}}+\alpha^2x\right]_\alpha^2+(a-2)(-\overset{2}{\cancel{4}}\alpha+2a\alpha)\frac{1}{\cancel{2}}$

$=\frac{8}{3}-4\alpha+2\alpha^2-\frac{1}{3}\alpha^3+\cancel{\alpha^3}-\cancel{\alpha^3}+(a-2)\alpha(a-2)$

$=-\frac{\alpha^3-6\alpha^2+12\alpha-8}{3}+(a-2)^2\alpha=-\frac{(\alpha-2)^3}{3}+(a-2)^2\alpha$

$$\begin{array}{r}
\alpha^2-4\alpha+4 \\
\alpha-2\,\big)\,\overline{\alpha^3-6\alpha^2+12\alpha-8} \\
\alpha^3-2\alpha^2 \\
\hline
-4\alpha^2+12\alpha \\
-4\alpha^2+8\alpha \\
\hline
4\alpha-8 \\
4\alpha-8 \\
\hline
0
\end{array}$$

ここで $\alpha-2=a-2-\sqrt{a^2-4}$

$(\alpha-2)^3=(a-2)^3-3(a-2)^2\sqrt{a^2-4}+3(a-2)(a-2)(a+2)-(a-2)(a+2)\sqrt{a^2-4}$

$=(a-2)^2(a-2+3a+6)+(a-2)\sqrt{a^2-4}(-3a+6-a-2)$

$=4(a-2)^2(a+1)-4(a-2)\sqrt{a^2-4}(a-1)$ より

$S=-\frac{4}{3}(a-2)^2(a+1)+\frac{4}{3}(a-2)(a-1)\sqrt{a^2-4}+a(a-2)^2-(a-2)^2\sqrt{a^2-4}$

$=\left(-\frac{4}{3}a-\frac{4}{3}+a\right)(a-2)^2+(a-2)\sqrt{a^2-4}\left(\frac{4}{3}a-\frac{4}{3}-a+2\right)$

$=\left(-\frac{1}{3}a-\frac{4}{3}\right)(a-2)^2+(a-2)\sqrt{a^2-4}\left(\frac{1}{3}a+\frac{2}{3}\right)=-\frac{1}{3}(a-2)^2(a+4)+\frac{1}{3}(a-2)(a+2)\sqrt{a^2-4}$