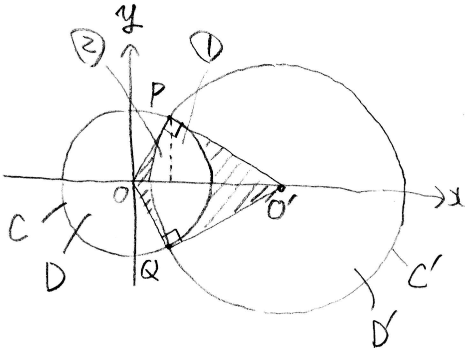
京大理系 1986前期 ⑥



P,Qの座標を$(r\cos\theta, r\sin\theta)$,$(r\cos\theta, -r\sin\theta)$ $(0<\theta<\frac{\pi}{2})$とおく

$\overrightarrow{OP}=(r\cos\theta, r\sin\theta)$ $\overrightarrow{O'P}=(r\cos\theta-1, r\sin\theta)$

$OP\perp O'P$ より $\overrightarrow{OP}\cdot\overrightarrow{O'P}=0$, $r^2\cos^2\theta-r\cos\theta+r^2\sin^2\theta=0$, $\cos\theta=r$

C'の半径は $|\overrightarrow{O'P}|=\sqrt{\cos^4\theta-2\cos^2\theta+1+\cos^2\theta(1-\cos^2\theta)}=\sqrt{1-\cos^2\theta}=\sin\theta$

左図の①の面積は $\pi\cos^2\theta\frac{\theta}{2\pi}-\cos^2\theta\cos\theta\sin\theta\frac{1}{2}$

$=\frac{1}{2}\theta\cos^2\theta-\frac{1}{2}\sin\theta\cos^3\theta$

左図の②の面積は $\pi\sin^2\theta\frac{\frac{\pi}{2}-\theta}{2\pi}-(1-\cos^2\theta)\cos\theta\sin\theta\frac{1}{2}$

$=\frac{1}{4}\pi\sin^2\theta-\frac{1}{2}\theta\sin^2\theta-\frac{1}{2}\sin^3\theta\cos\theta$

求める面積は上図の斜線部の面積であるから $2(\cos\theta\sin\theta\frac{1}{2}-\frac{1}{2}\theta\cos^2\theta+\frac{1}{2}\sin\theta\cos^3\theta-\frac{1}{4}\pi\sin^2\theta+\frac{1}{2}\theta\sin^2\theta+\frac{1}{2}\sin^3\theta\cos\theta)$

$=\sin\theta\cos\theta-\theta\cos^2\theta+\sin\theta\cos^3\theta-\frac{1}{2}\pi\sin^2\theta+\theta\sin^2\theta+\sin^3\theta\cos\theta$

$=2\sin\theta\cos\theta-\theta\cos^2\theta-\frac{1}{2}\pi\sin^2\theta+\theta\sin^2\theta$

$f(\theta)=2\sin\theta\cos\theta-\theta\cos^2\theta-\frac{1}{2}\pi\sin^2\theta+\theta\sin^2\theta$ $(0<\theta<\frac{\pi}{2})$とおく

$f'(\theta)=2\cos\theta\cos\theta-2\sin\theta\sin\theta-\cos^2\theta+\theta\cdot2\cos\theta\sin\theta-\pi\sin\theta\cos\theta+\sin^2\theta+\theta\,2\sin\theta\cos\theta$

$\quad=\cos^2\theta-\sin^2\theta+(4\theta-\pi)\sin\theta\cos\theta$

$g(\theta)=-\sin^2\theta+\cos^2\theta+(4\theta-\pi)\sin\theta\cos\theta$ $(0<\theta<\frac{\pi}{2})$とおく

$g'(\theta)=\cancel{-2\sin\theta\cos\theta}\cancel{-2\cos\theta\sin\theta}\cancel{+4\sin\theta\cos\theta}+(4\theta-\pi)(\cos\theta\cos\theta-\sin\theta\sin\theta)$

$\quad=(4\theta-\pi)(\cos^2\theta-\sin^2\theta)$

$g'(\theta)=0$ のとき, $4\theta-\pi=0$ または $\cos^2\theta-\sin^2\theta=0$, $\theta=\frac{\pi}{4}$

θ	$\cdots$	$\frac{\pi}{4}$	$\cdots$
$g'(\theta)$	$-$	0	$-$
$g(\theta)$	↘	0	↘

$g(\theta)$の増減表は左表

よって $f(\theta)$の増減表は右表

θ	$\cdots$	$\frac{\pi}{4}$	$\cdots$
$f'(\theta)$	$+$	0	$-$
$f(\theta)$	↗	$1-\frac{\pi}{4}$	↘

※ $f(\frac{\pi}{4})=2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}-\cancel{\frac{\pi}{4}\frac{1}{2}}-\frac{1}{2}\pi\frac{1}{2}+\cancel{\frac{\pi}{4}\frac{1}{2}}=1-\frac{\pi}{4}$

ゆえに求める値は $1-\frac{\pi}{4}$