京大理系　2001前期　⑥

$$\int_0^{n\pi} e^{-x}|\sin nx|\,dx=\sum_{k=1}^{n^2}\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}|\sin nx|\,dx \quad \text{—①}$$

$$\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}\sin nx\,dx=\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}(-e^{-x})'\sin nx\,dx=-\left[e^{-x}\sin nx\right]_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}+\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}\cos nx\cdot n\,dx$$

$$=n\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}(-e^{-x})'\cos nx\,dx=-n\left[e^{-x}\cos nx\right]_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}+n\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}(-\sin nx)n\,dx$$

$$=-n\left\{e^{-\frac{k\pi}{n}}\cos k\pi-e^{-\frac{(k-1)\pi}{n}}\cos(k-1)\pi\right\}-n^2\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}\sin nx\,dx \text{ より}$$

$$\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}\sin nx\,dx=\frac{n}{n^2+1}\left\{-e^{-\frac{k\pi}{n}}\cos k\pi+e^{-\frac{(k-1)\pi}{n}}\cos(k-1)\pi\right\}=\begin{cases}\dfrac{n}{n^2+1}\left\{e^{-\frac{k\pi}{n}}+e^{-\frac{(k-1)\pi}{n}}\right\} & (k\text{が奇数のとき})\\[2ex] \dfrac{n}{n^2+1}\left\{-e^{-\frac{k\pi}{n}}-e^{-\frac{(k-1)\pi}{n}}\right\} & (k\text{が偶数のとき})\end{cases}$$

kが奇数のとき，$\displaystyle\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}|\sin nx|\,dx=\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}\sin nx\,dx=\frac{n}{n^2+1}\left\{e^{-\frac{k\pi}{n}}+e^{-\frac{(k-1)\pi}{n}}\right\}$

kが偶数のとき，$\displaystyle\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}|\sin nx|\,dx=-\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} e^{-x}\sin nx\,dx=\frac{n}{n^2+1}\left\{e^{-\frac{k\pi}{n}}+e^{-\frac{(k-1)\pi}{n}}\right\}$

$$\text{よって. ①}=\frac{n}{n^2+1}(1+e^{\frac{\pi}{n}})\sum_{k=1}^{n^2}\left(e^{-\frac{\pi}{n}}\right)^k=\frac{n}{n^2+1}(1+e^{\frac{\pi}{n}})\frac{e^{-\frac{\pi}{n}}(1-e^{-n\pi})}{1-e^{-\frac{\pi}{n}}}=\frac{n}{n^2+1}(1+e^{\frac{\pi}{n}})e^{-\frac{\pi}{n}}(1-e^{-n\pi})\frac{1}{\frac{e^{-\frac{\pi}{n}}-1}{-\frac{\pi}{n}}}\frac{1}{\frac{\pi}{n}}$$

$$=\frac{1}{\pi}\frac{1}{1+\frac{1}{n^2}}(1+e^{\frac{\pi}{n}})e^{-\frac{\pi}{n}}(1-e^{-n\pi})\frac{1}{\frac{e^m-1}{m}}$$

↓

$-\frac{\pi}{n}=m$とおく. $n\to\infty$のとき $m\to 0$

$$\text{ゆえに}\ \lim_{n\to\infty}\int_0^{n\pi} e^{-x}|\sin nx|\,dx=\lim_{n\to\infty}\frac{1}{\pi}\frac{1}{1+\frac{1}{n^2}}(1+e^{\frac{\pi}{n}})e^{-\frac{\pi}{n}}(1-e^{-n\pi})\frac{1}{\frac{e^m-1}{m}}=\frac{1}{\pi}1\cdot2\cdot1\cdot1\cdot1=\frac{2}{\pi}$$