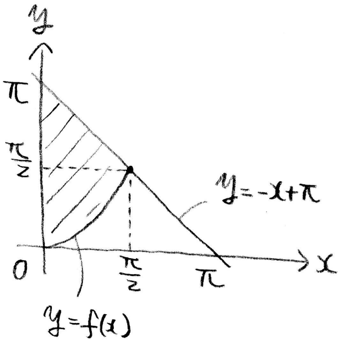
京大理系　2003前期　②

$f'(x)=\sin x+x\cos x$,　$0\leqq x\leqq \frac{\pi}{2}$ のとき, $f'(x)\geqq 0$,　$f(x)$ は単調増加

$f'(\frac{\pi}{2})=1$,　法線の方程式は　$y-\frac{\pi}{2}=-(x-\frac{\pi}{2})$,　$y=-x+\pi$



左図の斜線部を
x軸の回りに
回転して得られる
回転体の体積を
求めればよい.

求める体積を V とすると

$$\frac{V}{\pi}=\int_0^{\frac{\pi}{2}}(-x+\pi)^2dx-\int_0^{\frac{\pi}{2}}(x\sin x)^2dx=\int_0^{\frac{\pi}{2}}(x^2-2\pi x+\pi^2)dx-\int_0^{\frac{\pi}{2}}x^2\frac{1-\cos 2x}{2}dx$$

$$=\int_0^{\frac{\pi}{2}}(\frac{1}{2}x^2-2\pi x+\pi^2)dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}x^2\cos 2x\,dx$$

ここで $\int_0^{\frac{\pi}{2}}x^2\cos 2x\,dx=\int_0^{\frac{\pi}{2}}x^2\left(\frac{\sin 2x}{2}\right)'dx=\cancel{\left[x^2\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}}-\int_0^{\frac{\pi}{2}}\cancel{2}x\frac{\sin 2x}{\cancel{2}}dx$

$=\int_0^{\frac{\pi}{2}}x\left(\frac{\cos 2x}{2}\right)'dx=\left[x\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{\cos 2x}{2}dx=\frac{\pi}{4}(-1)-\frac{1}{2}\cancel{\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}}=-\frac{\pi}{4}$ より

$$\frac{V}{\pi}=\left[\frac{1}{2}\frac{x^3}{3}-\cancel{2}\pi\frac{x^2}{\cancel{2}}+\pi^2x\right]_0^{\frac{\pi}{2}}-\frac{\pi}{8}=\frac{\pi^3}{48}-\frac{\pi^3}{4}+\frac{\pi^3}{2}-\frac{\pi}{8}=\frac{1-12+24}{48}\pi^3-\frac{\pi}{8}=\frac{13}{48}\pi^3-\frac{\pi}{8}$$

$$V=\frac{13}{48}\pi^4-\frac{\pi^2}{8}$$