京大理系　2004前期　②

$g(x)=\dfrac{e}{x^a}-1\ (x>0)$ とする.

$g'(x)=-\dfrac{ae}{x^{a+1}}<0$, $g(x)$は単調減少

$g(x)=0$のとき. $\dfrac{e}{x^a}=1$. $x^a=e$. $x=e^{\frac{1}{a}}$

$h(x)=\dfrac{\log x}{x}\ (x>0)$ とする.

$h'(x)=\dfrac{\frac{1}{x}x-\log x}{x^2}=\dfrac{1-\log x}{x^2}$, $h'(x)=0$のとき. $\log x=1$. $x=e$

x	$\cdots$	e	$\cdots$
$h'(x)$	$+$	0	$-$
$h(x)$	$\nearrow$	$\frac{1}{e}$	$\searrow$

$h(x)$の増減表は左表のようになる.

$h(x)=0$ のとき. $x=1$

x	$\cdots$	1	$\cdots$	$e^{\frac{1}{a}}$	$\cdots$
$g(x)$	$+$	$+$	$+$	0	$-$
$h(x)$	$-$	0	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$-$

左表より. 求める面積をSとすると.

$$S=\int_1^{e^{\frac{1}{a}}}\left(\frac{e}{x^a}-1\right)\frac{\log x}{x}dx=e\int_1^{e^{\frac{1}{a}}}\frac{\log x}{x^{a+1}}dx-\int_1^{e^{\frac{1}{a}}}\frac{\log x}{x}dx$$

$$\int_1^{e^{\frac{1}{a}}}\frac{\log x}{x^{a+1}}dx=\int_1^{e^{\frac{1}{a}}}\log x\left(-\frac{1}{ax^a}\right)'dx=-\frac{1}{a}\left[\frac{\log x}{x^a}\right]_1^{e^{\frac{1}{a}}}+\frac{1}{a}\int_1^{e^{\frac{1}{a}}}\frac{1}{x^{a+1}}dx=-\frac{1}{a}\frac{\frac{1}{a}}{e}+\frac{1}{a}\left[-\frac{1}{ax^a}\right]_1^{e^{\frac{1}{a}}}$$

$(x^{-a})'=-a\dfrac{1}{x^{a+1}}$ より $\dfrac{1}{x^{a+1}}=\left(-\dfrac{1}{ax^a}\right)'$

$$=-\frac{1}{a^2e}-\frac{1}{a^2}\left(\frac{1}{e}-1\right)=-\frac{2}{a^2e}+\frac{1}{a^2}.$$

$$\int_1^{e^{\frac{1}{a}}}\frac{\log x}{x}dx=\int_1^{e^{\frac{1}{a}}}\log x(\log x)'dx=\int_1^{e^{\frac{1}{a}}}\log x\, d(\log x)=\left[\frac{(\log x)^2}{2}\right]_1^{e^{\frac{1}{a}}}=\frac{1}{2}\frac{1}{a^2}.$$

より. $S=-\dfrac{2}{a^2}+\dfrac{e}{a^2}-\dfrac{1}{2a^2}=\left(e-\dfrac{5}{2}\right)\dfrac{1}{a^2}$