京大理系　2004後期　①

$4nx(1-x)$ は $x=0,1$ で 0 となり $x=\frac{1}{2}$ に関して対称

$4nx(1-x)=1$ のとき, $4nx^2-4nx+1=0$ $\quad x=\frac{2n\pm\sqrt{4n^2-4n}}{4n}=\frac{n\pm\sqrt{n^2-n}}{2n}$

よって $\displaystyle\lim_{n\to+\infty} n\int_0^1 f(4nx(1-x))\,dx=\lim_{n\to+\infty} n\cdot 2\int_0^{\frac{n-\sqrt{n^2-n}}{2n}} 4nx(1-x)\,dx=\lim_{n\to+\infty} 8n^2\left[-\frac{x^3}{3}+\frac{x^2}{2}\right]_0^{\frac{n-\sqrt{n^2-n}}{2n}}$

$$=\lim_{n\to+\infty} 8n^2\left(-\frac{n^3-3n^2\sqrt{n^2-n}+3n(n^2-n)-(n^2-n)\sqrt{n^2-n}}{24n^3}+\frac{n^2-2n\sqrt{n^2-n}+n^2-n}{8n^2}\right)$$

$$=\lim_{n\to+\infty}\left(-\frac{4n^3-3n^2+(-4n^2+n)\sqrt{n^2-n}}{3n}+2n^2-n-2n\sqrt{n^2-n}\right)$$

$$=\lim_{n\to+\infty}\left(-\frac{4}{3}n^2+\cancel{n}+\left(\frac{4}{3}n-\frac{1}{3}\right)\sqrt{n^2-n}+2n^2-\cancel{n}-2n\sqrt{n^2-n}\right)=\lim_{n\to+\infty}\left(\frac{2}{3}n^2-\left(\frac{2}{3}n+\frac{1}{3}\right)\sqrt{n^2-n}\right)$$

$$=\lim_{n\to+\infty}\frac{1}{3}\left(2n^2-(2n+1)\sqrt{n^2-n}\right)=\lim_{n\to+\infty}\frac{1}{3}\frac{\left(2n^2-(2n+1)\sqrt{n^2-n}\right)\left(2n^2+(2n+1)\sqrt{n^2-n}\right)}{2n^2+(2n+1)\sqrt{n^2-n}}$$

$$=\lim_{n\to+\infty}\frac{1}{3}\frac{4n^4-(4n^2+4n+1)(n^2-n)}{2n^2+(2n+1)\sqrt{n^2-n}}=\lim_{n\to+\infty}\frac{1}{3}\frac{\cancel{4n^4}-\cancel{4n^4}+\cancel{4n^3}-\cancel{4n^3}+4n^2-n^2+n}{2n^2+(2n+1)\sqrt{n^2-n}}$$

$$=\lim_{n\to+\infty}\frac{1}{3}\frac{3+\frac{1}{n}}{2+\left(2+\frac{1}{n}\right)\sqrt{1-\frac{1}{n}}}=\frac{1}{4}$$