東工大　2008前期 ③

(1) 1.2.3.4.5.6 が出る確率を $\frac{1}{6}+a,\ \frac{1}{6}+b,\ \frac{1}{6}+c,\ \frac{1}{6}+d,\ \frac{1}{6}+e,\ \frac{1}{6}+f$ とする

$$P=\left(\tfrac{1}{6}+a\right)^2+\left(\tfrac{1}{6}+b\right)^2+\left(\tfrac{1}{6}+c\right)^2+\left(\tfrac{1}{6}+d\right)^2+\left(\tfrac{1}{6}+e\right)^2+\left(\tfrac{1}{6}+f\right)^2$$

$$=a^2+b^2+c^2+d^2+e^2+f^2+\tfrac{1}{3}(a+b+c+d+e+f)+\tfrac{1}{6}$$

ここで $\left(\frac{1}{6}+a\right)+\left(\frac{1}{6}+b\right)+\left(\frac{1}{6}+c\right)+\left(\frac{1}{6}+d\right)+\left(\frac{1}{6}+e\right)+\left(\frac{1}{6}+f\right)=1$ より $a+b+c+d+e+f=0$

$$P=a^2+b^2+c^2+d^2+e^2+f^2+\tfrac{1}{6}$$

よって $P\geqq\frac{1}{6}$ であり, 等号が成立するための必要十分条件は $a=b=c=d=e=f=0$

(2) $Q=\left\{\left(\frac{1}{6}+a\right)+\left(\frac{1}{6}+c\right)+\left(\frac{1}{6}+e\right)\right\}\left\{\left(\frac{1}{6}+b\right)+\left(\frac{1}{6}+d\right)+\left(\frac{1}{6}+f\right)\right\}=\left(\frac{1}{2}+a+c+e\right)\left(\frac{1}{2}+b+d+f\right)$

$$=\tfrac{1}{4}+\tfrac{1}{2}(b+d+f)+\tfrac{1}{2}(a+c+e)+(a+c+e)(b+d+f)=\tfrac{1}{4}-(a+c+e)^2$$

よって $\frac{1}{4}\geqq Q$

$$Q-\tfrac{1}{2}+\tfrac{3}{2}P=\cancel{\tfrac{1}{4}}+(a+c+e)(b+d+f)-\cancel{\tfrac{1}{2}}+\tfrac{3}{2}(a^2+b^2+c^2+d^2+e^2+f^2)+\cancel{\tfrac{1}{4}}$$

$$=(a+c+e)(b+d+f)+2(a^2+b^2+c^2+d^2+e^2+f^2)-\tfrac{1}{2}(a^2+b^2+c^2+d^2+e^2+f^2)$$

$$=ab+ad+af+bc+cd+cf+be+de+ef+2(a^2+b^2+c^2+d^2+e^2+f^2)-\tfrac{1}{2}\{(a+c+e)^2-2ac-2ae-2ce+(b+d+f)^2-2bd-2bf-2df\}$$

$$=ab+ad+af+bc+cd+cf+be+de+ef+ac+ae+ce+bd+bf+df+2(a^2+b^2+c^2+d^2+e^2+f^2)-\tfrac{1}{2}(a+c+e)^2-\tfrac{1}{2}(b+d+f)^2$$

ここで $0=(a+b+c+d+e+f)^2$

$=a^2+b^2+c^2+d^2+e^2+f^2+2ab+2ac+2ad+2ae+2af+2bc+2bd+2be+2bf+2cd+2ce+2cf+2de+2df+2ef$ より

$$Q-\tfrac{1}{2}+\tfrac{3}{2}P=\tfrac{3}{2}(a^2+b^2+c^2+d^2+e^2+f^2)-\tfrac{1}{2}a^2-\tfrac{1}{2}c^2-\tfrac{1}{2}e^2-ac-ae-ce-\tfrac{1}{2}b^2-\tfrac{1}{2}d^2-\tfrac{1}{2}f^2-bd-bf-df$$

$$=a^2+b^2+c^2+d^2+e^2+f^2-ac-ce-ea-bd-df-fb$$

$$=\tfrac{1}{2}(a-c)^2+\tfrac{1}{2}(c-e)^2+\tfrac{1}{2}(e-a)^2+\tfrac{1}{2}(b-d)^2+\tfrac{1}{2}(d-f)^2+\tfrac{1}{2}(f-b)^2\geqq 0$$

よって $Q\geqq\frac{1}{2}-\frac{3}{2}P$