東工大　2008後期　②

$$I_n=\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} x^2|\sin n\pi x|\,dx=\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} x^2\sin n\pi\left(x-\frac{k}{n}\right)dx$$

ここで $\displaystyle\int_{\frac{k}{n}}^{\frac{k+1}{n}} x^2\sin n\pi\left(x-\frac{k}{n}\right)dx=\int_{\frac{k}{n}}^{\frac{k+1}{n}} x^2\left\{-\frac{\cos n\pi\left(x-\frac{k}{n}\right)}{n\pi}\right\}'dx=\left[-\frac{\cos n\pi\left(x-\frac{k}{n}\right)}{n\pi}x^2\right]_{\frac{k}{n}}^{\frac{k+1}{n}}+\int_{\frac{k}{n}}^{\frac{k+1}{n}} 2x\frac{\cos n\pi\left(x-\frac{k}{n}\right)}{n\pi}dx$

$$=\frac{1}{n\pi}\frac{k^2+2k+1}{n^2}+\frac{1}{n\pi}\frac{k^2}{n^2}+\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{2x}{n\pi}\left\{\frac{\sin n\pi\left(x-\frac{k}{n}\right)}{n\pi}\right\}'dx=\frac{2k^2+2k+1}{n^3\pi}+\cancel{\left[\frac{2x}{n^2\pi^2}\sin n\pi\left(x-\frac{k}{n}\right)\right]_{\frac{k}{n}}^{\frac{k+1}{n}}}-\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{2}{n^2\pi^2}\sin n\pi\left(x-\frac{k}{n}\right)dx$$

$$=\frac{2k^2+2k+1}{n^3\pi}+\frac{2}{n^2\pi^2}\left[\frac{\cos n\pi\left(x-\frac{k}{n}\right)}{n\pi}\right]_{\frac{k}{n}}^{\frac{k+1}{n}}=\frac{2k^2+2k+1}{n^3\pi}+\frac{2}{n^3\pi^3}(-1-1)=\frac{2k^2+2k+1}{n^3\pi}-\frac{4}{n^3\pi^3}$$ より

$$I_n=\frac{1}{n^3\pi}\left[2\frac{1}{\cancel{6}_{3}}(n-1)n\{2(n-1)+1\}+\cancel{2}\frac{1}{\cancel{2}}(n-1)n+n\right]-\frac{4}{n^2\pi^3}$$

$$=\frac{1}{n^3\pi}\frac{(n^2-n)(2n-1)+3n^2}{3}-\frac{4}{n^2\pi^3}=\frac{1}{n^3\pi}\frac{2n^3+n}{3}-\frac{4}{n^2\pi^3}=\frac{2}{3\pi}+\frac{1}{3n^2\pi}-\frac{4}{n^2\pi^3}$$

$$\lim_{n\to\infty}I_n=\frac{2}{3\pi}$$