東大理科　1962前期　④

$$\begin{cases} \dfrac{1}{1-r}-\dfrac{1}{1+r}=\dfrac{1+r-1+r}{1-r^2}=\dfrac{2r}{1-r^2} \\ \dfrac{1}{1-r}+\dfrac{1}{1+r}=\dfrac{1+r+1-r}{1-r^2}=\dfrac{2}{1-r^2} \end{cases}$$

$$\frac{2r}{1-r^2}=\frac{1}{1-r}-\frac{1}{1+r}=\frac{2}{1-r^2}-\frac{2}{1+r}$$

$$\begin{cases} \dfrac{1}{1-r^2}-\dfrac{1}{1+r^2}=\dfrac{1+r^2-1+r^2}{1-r^4}=\dfrac{2r^2}{1-r^4} \\ \dfrac{1}{1-r^2}+\dfrac{1}{1+r^2}=\dfrac{1+r^2+1-r^2}{1-r^4}=\dfrac{2}{1-r^4} \end{cases}$$

$$\frac{2r^2}{1-r^4}=\frac{1}{1-r^2}-\frac{1}{1+r^2}=\frac{2}{1-r^4}-\frac{2}{1+r^2}$$

$$\begin{cases} \dfrac{1}{1-r^4}-\dfrac{1}{1+r^4}=\dfrac{1+r^4-1+r^4}{1-r^8}=\dfrac{2r^4}{1-r^8} \\ \dfrac{1}{1-r^4}+\dfrac{1}{1+r^4}=\dfrac{1+r^4+1-r^4}{1-r^8}=\dfrac{2}{1-r^8} \end{cases}$$

$$\frac{2r^4}{1-r^8}=\frac{1}{1-r^4}-\frac{1}{1+r^4}=\frac{2}{1-r^8}-\frac{2}{1+r^4}$$

$S_n=\dfrac{r}{1-r^2}+\dfrac{r^2}{1-r^4}+\dfrac{r^4}{1-r^8}+\cdots+\dfrac{r^{2^{n-1}}}{1-r^{2^n}}$ とおく.

$$S_n=\frac{1}{1-r^2}-\frac{1}{1+r}+\frac{1}{1-r^4}-\frac{1}{1+r^2}+\frac{1}{1-r^8}-\frac{1}{1+r^4}+\cdots+\frac{1}{1-r^{2^n}}-\frac{1}{1+r^{2^{n-1}}}$$

$$=\frac{2r^2}{1-r^4}+\frac{2r^4}{1-r^8}+\cdots+\frac{2r^{2^{n-1}}}{1-r^{2^n}}-\frac{1}{1+r}+\frac{1}{1-r^{2^n}}$$

$$=2S_n-\frac{2r}{1-r^2}-\frac{1}{1+r}+\frac{1}{1-r^{2^n}}$$

$$S_n=\frac{1}{1-r}-\cancel{\frac{1}{1+r}}+\cancel{\frac{1}{1+r}}-\frac{1}{1-r^{2^n}}=\frac{1}{1-r}-\frac{1}{1-r^{2^n}}$$

よって求める和を $S$ とすると

$|r|>1$ のとき $S=\lim_{n\to\infty}S_n=\dfrac{1}{1-r}$

$|r|<1$ のとき $S=\lim_{n\to\infty}S_n=\dfrac{1}{1-r}-1=\dfrac{1-1+r}{1-r}=\dfrac{r}{1-r}$