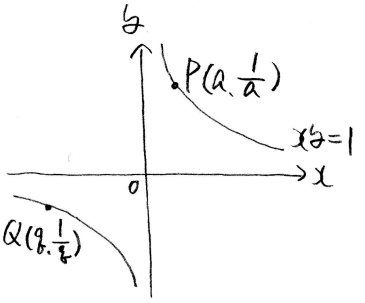
東大理科　1969前期　⑤




(1) Qの座標を $(g, \frac{1}{g})$ $(g<0)$ とおく.

Pの座標は $(a, \frac{1}{a})$ とおける.

QPの長さの2乗は $(g-a)^2+(\frac{1}{g}-\frac{1}{a})^2$

$f(g)=g^2-2ag+a^2+\frac{1}{g^2}-\frac{2}{ag}+\frac{1}{a^2}$ $(g<0)$ とおく.

$f'(g)=2g-2a-\frac{2}{g^3}+\frac{2}{ag^2}=2(g-a)-\frac{2}{g^2}(\frac{1}{g}-\frac{1}{a})$

$=2(g-a)-\frac{2}{g^2}\frac{a-g}{ga}=2(g-a)(1+\frac{1}{ag^3})$

$f'(g)=0$ のとき $g^3=-\frac{1}{a}$, $g=-\frac{1}{a^{\frac{1}{3}}}$

| $g$ | … | $-\frac{1}{a^{\frac{1}{3}}}$ | … |
|---|---|---|---|
| $f'(g)$ | − | 0 | + |
| $f(g)$ | ↘ | $(a^{\frac{2}{3}}+a^{-\frac{2}{3}})^{\frac{3}{2}}$ | ↗ |

$f(g)$ の増減表は左表のようになる.

よってQPの長さの最小値は $(a^{\frac{2}{3}}+a^{-\frac{2}{3}})^{\frac{3}{2}}$

$a^{\frac{1}{3}}=\alpha$ とおくと

$f(-\frac{1}{\alpha})=\frac{1}{\alpha^2}+2\alpha^3\frac{1}{\alpha}+\alpha^6+\alpha^2+\frac{2}{\alpha^3}\alpha+\frac{1}{\alpha^6}$

$=\frac{1}{\alpha^2}+2\alpha^2+\alpha^6+\alpha^2+\frac{2}{\alpha^2}+\frac{1}{\alpha^6}$

$=\alpha^6+3\alpha^2+\frac{3}{\alpha^2}+\frac{1}{\alpha^6}=\frac{\alpha^{12}+3\alpha^8+3\alpha^4+1}{\alpha^6}$

$=\frac{(\alpha^4+1)^3}{\alpha^6}=(\alpha^2+\frac{1}{\alpha^2})^3=(a^{\frac{2}{3}}+a^{-\frac{2}{3}})^3$

(2) $\frac{\frac{1}{a}+a^{\frac{1}{3}}}{a+\frac{1}{a^{\frac{1}{3}}}}=\frac{1}{\sqrt{3}}$ であればよいから

$a^{\frac{1}{3}}=\alpha$ とおくと

$\frac{\frac{1}{\alpha^3}+\alpha}{\alpha^3+\frac{1}{\alpha}}=\frac{1}{\sqrt{3}}$, $\frac{\alpha^4+1}{\alpha^6+\alpha^2}=\frac{1}{\sqrt{3}}$ $\alpha^6-\sqrt{3}\alpha^4+\alpha^2-\sqrt{3}=0$

$\alpha^4(\alpha^2-\sqrt{3})+\alpha^2-\sqrt{3}=0$, $(\alpha^4+1)(\alpha^2-\sqrt{3})=0$,

$a^{\frac{2}{3}}=3^{\frac{1}{2}}$, $a=3^{\frac{3}{4}}$