東大理科　1972前期　⑤

$$u(x)=\int_0^x h(t)f(t)\,dt-h(x)\int_1^x f(t)\,dt$$

$$\frac{du}{dx}=\cancel{h(x)f(x)}-h'(x)\int_1^x f(t)\,dt-\cancel{h(x)f(x)}$$

$$\frac{d^2u}{dx^2}=-h''(x)\int_1^x f(t)\,dt-h'(x)f(x)$$

$f(x)=ax+b$ とおくと (1)より.

$$-h''(x)\left[a\frac{t^2}{2}+bt\right]_1^x-h'(x)(ax+b)=ax+b,\quad -h''(x)\left(a\frac{x^2}{2}+bx-\frac{a}{2}-b\right)-h'(x)(ax+b)=ax+b$$

$$a\left\{-\frac{1}{2}x^2h''(x)+\frac{1}{2}h''(x)-xh'(x)-x\right\}+b\left\{-xh''(x)+h''(x)-h'(x)-1\right\}=0.$$

$$\begin{cases}(x^2-1)h''(x)+2xh'(x)=-2x\\(x-1)h''(x)+h'(x)=-1\end{cases}$$

$$\begin{array}{rl} & (x^2-1)h''(x)+2xh'(x)=-2x\\ -) & 2x(x-1)h''(x)+2xh'(x)=-2x\\ \hline & (x^2-1-2x^2+2x)h''(x)=0.\end{array}$$

$(x-1)^2h''(x)=0.$　$h''(x)=0$

$h'(x)=-1.$

$h(x)=-x+C.$

(2)より.　$C\int_0^1 f(t)\,dt=0.$　$C=0.$

よって　$h(x)=-x$