東大理科　1975前期　④

$2\sin\theta_{n+1}=\sqrt{2+2\sin\theta_n}$

$4\sin^2\theta_{n+1}=2+2\sin\theta_n$

$4\dfrac{1-\cos 2\theta_{n+1}}{2}=2+2\sin\theta_n$

$-\cos 2\theta_{n+1}=\sin\theta_n$

ここで $\sin\left(2\theta_{n+1}-\dfrac{\pi}{2}\right)=-\cos 2\theta_{n+1}\cdot\sin\dfrac{\pi}{2}=-\cos 2\theta_{n+1}$ であるから

$\sin\left(2\theta_{n+1}-\dfrac{\pi}{2}\right)=\sin\theta_n$

$2\theta_{n+1}-\dfrac{\pi}{2}=\theta_n$,　$\theta_{n+1}=\dfrac{1}{2}\theta_n+\dfrac{\pi}{4}$　　　　$x=\dfrac{1}{2}x+\dfrac{\pi}{4}$　$x=\dfrac{\pi}{2}$

$\theta_{n+1}-\dfrac{\pi}{2}=\dfrac{1}{2}\left(\theta_n-\dfrac{\pi}{2}\right)$

$\theta_n-\dfrac{\pi}{2}=\dfrac{1}{2}\left(\theta_{n-1}-\dfrac{\pi}{2}\right)=\left(\dfrac{1}{2}\right)^2\left(\theta_{n-2}-\dfrac{\pi}{2}\right)=\cdots=\left(\dfrac{1}{2}\right)^{n-1}\left(\theta_1-\dfrac{\pi}{2}\right)$

ここで $\sqrt{2}=2\sin\theta_1$　$\theta_1=\dfrac{\pi}{4}$ であるから

$\theta_n=\left(\dfrac{1}{2}\right)^{n-1}\left(-\dfrac{\pi}{4}\right)+\dfrac{\pi}{2}=\dfrac{\pi}{2}\left\{1-\left(\dfrac{1}{2}\right)^n\right\}$

$\lim_{n\to\infty}\theta_n=\dfrac{\pi}{2}$