東大理科 1998後期 ②

(1) $\int_0^{2\pi}\cancel{\sin^2x}\,dx-2\int_0^{2\pi}f(x)\sin x\,dx+\int_0^{2\pi}f(x)^2dx=\int_0^{2\pi}\cancel{\sin^2x}\,dx$

$\int_0^{2\pi}f(x)\sin x\,dx=r\int_0^{2\pi}\sin(x+\theta)\sin x\,dx=\frac{r}{2}\int_0^{2\pi}\{\cos\theta-\cos(2x+\theta)\}dx=\frac{r}{2}\left[x\cos\theta-\frac{1}{2}\sin(2x+\theta)\right]_0^{2\pi}=\pi r\cos\theta$ —①

※ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$-\underline{)\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$

$\int_0^{2\pi}f(x)^2dx=r^2\int_0^{2\pi}\sin^2(x+\theta)dx=r^2\int_0^{2\pi}\frac{1-\cos(2x+2\theta)}{2}dx$

$=\frac{r^2}{2}\left[x-\frac{1}{2}\sin(2x+2\theta)\right]_0^{2\pi}=\pi r^2$ —②

①②より $2\pi r\cos\theta=\pi r^2$. $r=2\cos\theta$

(2) θが0から$\frac{\pi}{4}$まで動くとき. rは2から$\sqrt{2}$まで動く.

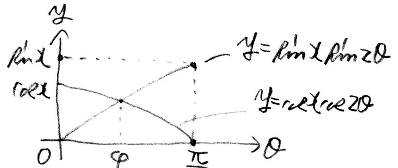
$g(\theta)=2\cos\theta\sin(x+\theta)$ $(0\leqq\theta\leqq\frac{\pi}{4})$ を考える

$g'(\theta)=2\{-\sin\theta\sin(x+\theta)+\cos\theta\cos(x+\theta)\}=2\{-\sin\theta(\sin x\cos\theta+\cos x\sin\theta)+\cos\theta(\cos x\cos\theta-\sin x\sin\theta)\}$

$=2\{-2\sin x\sin\theta\cos\theta+\cos x(\cos^2\theta-\sin^2\theta)\}=2(\cos x\cos2\theta-\sin x\sin2\theta)$

(i) $x=0$のとき. $g(\theta)=\sin2\theta$

(ii) $0<x<\frac{\pi}{2}$のとき. $\cos x>0$. $\sin x>0$より, 左下図より $g'(\theta)=0$ を満たすθはただ1つ存在するから これをφとすると



$g(\theta)$の増減表は右表

θ	0	$\cdots$	φ	$\cdots$	$\frac{\pi}{4}$
$g'(\theta)$		$+$	0	$-$	
$g(\theta)$	$2\sin x$	$\nearrow$	$\sin x+1$	$\searrow$	$\sin x+\cos x$

※ $g(\frac{\pi}{4})=\sqrt{2}(\sin x\frac{1}{\sqrt{2}}+\cos x\frac{1}{\sqrt{2}})=\sin x+\cos x$

※ $\cos x(1-2\sin^2\varphi)=2\sin x\sin\varphi\cos\varphi$ より.

$\cos^2x(1-4\sin^2\varphi+4\sin^4\varphi)=4\sin^2x\sin^2\varphi(1-\sin^2\varphi)$

$(4\sin^2x+4\cos^2x)\sin^4\varphi-(4\sin^2x+4\cos^2x)\sin^2\varphi+\cos^2x=0$

$\sin^4\varphi-\sin^2\varphi+\frac{\cos^2x}{4}=0$. $\sin^2\varphi=\frac{1\pm\sqrt{1-\cos^2x}}{2}=\frac{1\pm\sin x}{2}$

$0<\varphi<\frac{\pi}{4}$より. $0<\sin\varphi<\frac{1}{\sqrt{2}}$ $0<\sin^2\varphi<\frac{1}{2}$. $\sin^2\varphi=\frac{1-\sin x}{2}$. $\cos^2\varphi=\frac{1+\sin x}{2}$

$g(\varphi)=2\frac{\sqrt{1+\sin x}}{\sqrt{2}}\left(\sin x\frac{\sqrt{1+\sin x}}{\sqrt{2}}+\cos x\frac{\sqrt{1-\sin x}}{\sqrt{2}}\right)=\sin x(1+\sin x)+\cos x\cdot\cos x=\sin x+1$

※ $2\sin x=\sin x+\cos x$ のとき. $\sin x=\cos x$. $x=\frac{\pi}{4}$

(iii) $x=\frac{\pi}{2}$のとき. $g(\theta)=2\cos^2\theta$

(iv) $\frac{\pi}{2}<x<\pi$のとき.

$\cos x<0$. $\sin x>0$より. $g'(\theta)<0$

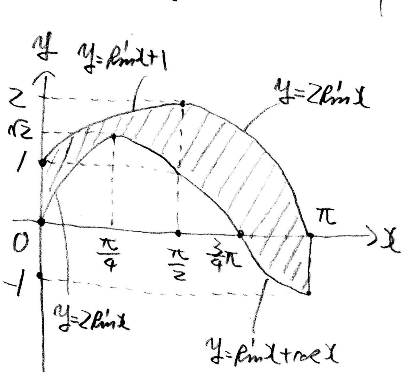
$g(\theta)$は単調減少

$g(0)=2\sin x$

$g(\frac{\pi}{4})=\sqrt{2}(\sin x\frac{1}{\sqrt{2}}+\cos x\frac{1}{\sqrt{2}})$

$=\sin x+\cos x$

(v) $x=\pi$のとき $g(\theta)=-\sin2\theta$



(i)〜(v)より Dは上図の斜線部

境界線上の点含む.

(3) 求める面積をSとすると.

$S=\int_0^{\frac{\pi}{4}}(-\sin x+1)dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(-\cos x+1)dx+\int_{\frac{\pi}{2}}^{\pi}(\sin x-\cos x)dx$

$=\left[\cos x+x\right]_0^{\frac{\pi}{4}}+\left[-\sin x+x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}+\left[-\cos x-\sin x\right]_{\frac{\pi}{2}}^{\pi}$

$=\frac{1}{\sqrt{2}}+\cancel{\frac{\pi}{4}}-1-\cancel{1}+\frac{\pi}{2}+\frac{1}{\sqrt{2}}-\cancel{\frac{\pi}{4}}+\cancel{1}+\cancel{1}=\frac{\pi}{2}+\sqrt{2}$