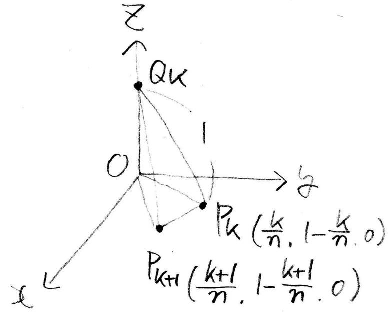
東大理科　2002前期　⑤




$Q_k$の座標を$(0, 0, z)$とおくと

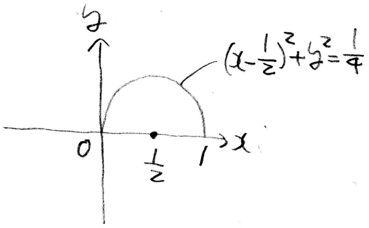
$\sqrt{(-\frac{k}{n})^2+(1-\frac{k}{n})^2+z^2}=1$，　$\frac{k^2}{n^2}+\cancel{1}-\frac{2k}{n}+\frac{k^2}{n^2}+z^2=\cancel{1}$

$z^2=-\frac{2k^2}{n^2}+\frac{2k}{n}$，$z^2=\frac{2k}{n}(1-\frac{k}{n})$，$z\geqq 0$より　$z=\sqrt{\frac{2k}{n}(1-\frac{k}{n})}$

$\triangle OP_kP_{k+1}$の面積は　$\frac{1}{2}\left|\frac{k}{n}(1-\frac{k+1}{n})-(1-\frac{k}{n})\frac{k+1}{n}\right|=\frac{1}{2}\left|\cancel{\frac{k}{n}}-\cancel{\frac{k^2}{n^2}}-\cancel{\frac{k}{n^2}}-\cancel{\frac{k}{n}}-\frac{1}{n}+\cancel{\frac{k^2}{n^2}}+\cancel{\frac{k}{n^2}}\right|=\frac{1}{2n}$　であるから

$V_k=\frac{1}{6n}\sqrt{\frac{2k}{n}(1-\frac{k}{n})}$．

$\lim_{n\to\infty}\sum_{k=0}^{n-1}V_k=\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\sqrt{2}}{6}\sqrt{\frac{k}{n}(1-\frac{k}{n})}\frac{1}{n}=\int_0^1\frac{\sqrt{2}}{6}\sqrt{x(1-x)}\,dx$




ここで，中心$(\frac{1}{2}, 0)$ 半径$\frac{1}{2}$，$y\geqq 0$である半円の方程式は

$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$，$y\geqq 0$，　$x^2-x+\cancel{\frac{1}{4}}+y^2=\cancel{\frac{1}{4}}$，$y\geqq 0$

$y^2=x(1-x)$，$y\geqq 0$，　$y=\sqrt{x(1-x)}$　であるから　$\int_0^1\sqrt{x(1-x)}\,dx=\frac{1}{2}\pi\frac{1}{4}=\frac{\pi}{8}$

$\lim_{n\to\infty}\sum_{k=0}^{n-1}V_k=\frac{\sqrt{2}\pi}{48}$