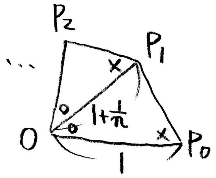
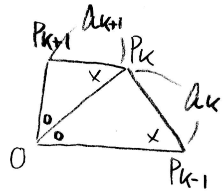
東大理科 2007前期 ②

$\angle P_0OP_1 = \angle P_1OP_2 = \cdots = \frac{\pi}{n}$

$\angle OP_1P_2 = \angle OP_2P_3 = \cdots = \angle OP_0P_1$





$\triangle OP_{k-1}P_k \sim \triangle OP_kP_{k+1}$ $(k=1, 2 \ldots n-1)$

$\frac{a_{k+1}}{a_k} = 1+\frac{1}{n}$ より a_k は 初項 a_1, 公比 $1+\frac{1}{n}$ の等比数列

$$S_n = \frac{a_1\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}}{1+\frac{1}{n}-1} = na_1\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}$$

余弦定理より $a_1^2 = 1+\left(1+\frac{1}{n}\right)^2 - 2\left(1+\frac{1}{n}\right)\cos\frac{\pi}{n} = 1+1+\frac{2}{n}+\frac{1}{n^2} - 2\left(1+\frac{1}{n}\right)\cos\frac{\pi}{n}$

$$= 2\left(1+\frac{1}{n}\right)\left(1-\cos\frac{\pi}{n}\right) + \frac{1}{n^2} = 4\left(1+\frac{1}{n}\right)\sin^2\frac{\pi}{2n} + \frac{1}{n^2}$$

$$= \frac{\pi^2}{n^2}\left(1+\frac{1}{n}\right)\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^2 + \frac{1}{n^2}$$ より

$$S_n = \sqrt{\pi^2\left(1+\frac{1}{n}\right)\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)^2 + 1}\;\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}$$

$$\lim_{n\to\infty} S_n = \sqrt{\pi^2+1}\,(e-1)$$