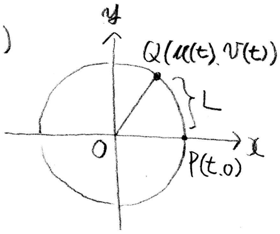
東大理科　2011前期　③

(1)



$L=\theta t$ とおくと

$u(t)=t\cos\theta=t\cos\frac{L}{t}$

$v(t)=t\sin\theta=t\sin\frac{L}{t}$

(2) $u'(t)=\cos\frac{L}{t}-\cancel{t}\sin\frac{L}{t}\cdot L\left(-\frac{1}{t^{\cancel{2}}}\right)=\cos\frac{L}{t}+\frac{L}{t}\sin\frac{L}{t}$　　$v'(t)=\sin\frac{L}{t}+\cancel{t}\cos\frac{L}{t}\cdot L\left(-\frac{1}{t^{\cancel{2}}}\right)=\sin\frac{L}{t}-\frac{L}{t}\cos\frac{L}{t}$

$\{u'(t)\}^2+\{v'(t)\}^2=\cos^2\frac{L}{t}+\cancel{2\frac{L}{t}\sin\frac{L}{t}\cos\frac{L}{t}}+\frac{L^2}{t^2}\sin^2\frac{L}{t}+\sin^2\frac{L}{t}-\cancel{2\frac{L}{t}\sin\frac{L}{t}\cos\frac{L}{t}}+\frac{L^2}{t^2}\cos^2\frac{L}{t}=1+\frac{L^2}{t^2}$

$f(a)=\int_a^1\sqrt{1+\frac{L^2}{t^2}}\,dt=\int_a^1\frac{\sqrt{t^2+L^2}}{t}dt=\int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}}\frac{x}{\sqrt{x^2-L^2}}\frac{x}{\sqrt{x^2-L^2}}dx=\int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}}\frac{x^2-L^2+L^2}{x^2-L^2}dx=\int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}}\left(1+\frac{L^2}{x^2-L^2}\right)dx$

$\sqrt{t^2+L^2}=x$ とおく　$\begin{array}{c|c} t & a\to 1 \\ \hline x & \sqrt{a^2+L^2}\to\sqrt{1+L^2}\end{array}$　$\frac{dx}{dt}=\frac{1}{\cancel{2}}\frac{\cancel{2}t}{\sqrt{t^2+L^2}}$

$=\int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}}\left(1+\frac{L}{2}\frac{1}{x-L}-\frac{L}{2}\frac{1}{x+L}\right)dx=\left[x+\frac{L}{2}\log(x-L)-\frac{L}{2}\log(x+L)\right]_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}}$

$\downarrow$

$\frac{1}{x-L}-\frac{1}{x+L}=\frac{x+L-(x-L)}{(x-L)(x+L)}=\frac{2L}{x^2-L^2}$ より　$\frac{L^2}{x^2-L^2}=\frac{L}{2}\left(\frac{1}{x-L}-\frac{1}{x+L}\right)$

$=\sqrt{1+L^2}-\sqrt{a^2+L^2}+\frac{L}{2}\log(\sqrt{1+L^2}-L)-\frac{L}{2}\log(\sqrt{1+L^2}+L)-\frac{L}{2}\log(\sqrt{a^2+L^2}-L)+\frac{L}{2}\log(\sqrt{a^2+L^2}+L)$

(3) $-\log(\sqrt{a^2+L^2}-L)+\log(\sqrt{a^2+L^2}+L)=\log\frac{\sqrt{a^2+L^2}+L}{\sqrt{a^2+L^2}-L}=\log\frac{(\sqrt{a^2+L^2}+L)^2}{(\sqrt{a^2+L^2}-L)(\sqrt{a^2+L^2}+L)}=\log\frac{(\sqrt{a^2+L^2}+L)^2}{a^2+\cancel{L^2}-\cancel{L^2}}$

$=2\log(\sqrt{a^2+L^2}+L)-2\log a$　より

$f(a)=\sqrt{1+L^2}-\sqrt{a^2+L^2}+\frac{L}{2}\log(\sqrt{1+L^2}-L)-\frac{L}{2}\log(\sqrt{1+L^2}+L)+L\log(\sqrt{a^2+L^2}+L)-L\log a$

$\lim_{a\to+0}\frac{f(a)}{\log a}=\lim_{a\to+0}\left\{\frac{\sqrt{1+L^2}-\sqrt{a^2+L^2}+\frac{L}{2}\log(\sqrt{1+L^2}-L)-\frac{L}{2}\log(\sqrt{1+L^2}+L)+L\log(\sqrt{a^2+L^2}+L)}{\log a}-L\right\}=-L$